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Nothing in Moderation, Everything in Excess: A New Weighted Statistic on Permutations

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Nothing in Moderation, Everything in Excess
(A New Weighted Statistic on Permutations)

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The major index is a well-known statistic on permutations which is computed by summing the positions of descents in a permutation. Here, instead of considering descents, I investigate what happens when excedances are weighted by position in a permutation. I present several theorems concerning various symmetries of the resulting distribution. I also offer a preliminary analysis of my attempts to generalize the results to multiset permutations.

Permutations can be described in a wide variety of ways, each of which is known as a statistic. One of the best known permutation statistics is the major index, which comes from weighting descents by position. A descent occurs in a permutation when a number is larger than the next number in the permutation. For instance in the permutation 41325, positions 1 and 3 contain descents. The weight of this permutation is 4, the sum total of the descent positions. Thus the major index of 41325 is 4. The distribution of the number of descents on all permutations of a set {1, 2, ..., n} is known as the Eulerian distribution (Stanley, 22); the distribution of the major index across that set of permutations is called the Mahonian distribution. The bivariate distribution of both the major index and descent also has a number of interesting properties which will not be discussed here.

Another statistic on permutations involves excedances--when a number exceeds its position. The excedance weight is found by summing the positions in which excedances occur. Our previous example, 41325, yields one excedance in the first position and a weight of 1. It is known that the distribution of the number of excedances is the same distribution as the number of descents, or the Eulerian distribution (Stanley, 23).

Since numbers of both excedances and descents lead to the Eulerian polynomial, an immediate question is whether excedance and descent weights give the same distribution. Looking at weights alone is unfruitful; the distribution of excedance weights is quite uninteresting. However, when we look at both the number of excedances and a weight, or a bivariate distribution, we get a distribution with a number of apparent properties (Appendix A). All permutations of {1, 2, ..., n} for $n \leq 8$ were processed through a computer program written in the C programming language on an IRIS Indigo workstation (Appendix B). Following the naming system of the Mahonian and Eulerian distributions, I propose to call this excedance and weight distribution the Paulukonis distribution.

Before delving into theorems about the Paulukonis distribution, some basic definitions and propositions are needed. Consider the permutation 413526. In two-line permutation notation this would be $\begin{smallmatrix}123456\\413526\end{smallmatrix}$; the top line indicates the position while the bottom number is the value of the permutation. As noted earlier, an excedance occurs whenever a number is greater than its position: in this case, 4 exceeds 1 and 5 exceeds 4. Any number which is equal to or less than its position is considered a non-excedance; in our example, that occurs in positions 2, 3, 5 and 6. Fixed points are those in which position and number are the same; note that some non-excedances are also fixed points. I may refer to "fixed points and non-excedances"; in this case, fixed points are not included with the non-excedances.

A permutation is weighted by summing the positions of the excedances. Thus 413526 has 2 excedances and a weight of 1 + 4 = 5. Notice that the number 1 can never exceed; any other number will always exceed in position 1. The number n will exceed any position (except n) while position n can never be exceeded.

In this paper, I will use a one line notation for permutations: $w_1w_2w_3...w_n$, where w_i is the number in position i. Let S_n denote the set of permutations on {1, 2, ..., n} and S(n,j,k) denote the subset of S_n consisting of permutations which have j excedances summing to weight k. Then N(n,j,k) denotes the number of elements in the set S(n,j,k). We let S(n,j) indicate the "excedance block" of all permutations in S_n with j excedances. Note that the identity permutation, 123...n, is the only element contained in S(n,0,0).

Now that the basic definitions and notation are set down, let us consider the first proposition. It may seem fairly obvious, but it is necessary to point out as it will be useful in the theorems.

Proposition 1: If x is the first position exceeded, positions 1, 2, ..., (x-1) are fixed.

Proof: Suppose to the contrary that position i < x is the first position such that $w_i \neq i$. Then $w_1 \ldots w_{i-1}$ must be fixed ($w_1 = 1$, etc.). Since x is the first position exceeded, w_i must be less than i; however, all of the numbers {1, ..., i-1} have already occurred in w. This is a contradiction, $\Rightarrow\Leftarrow$. Therefore, all positions below x are fixed. ∎

A natural question regarding the numbers N(n,j,k) is: Can the weight for a given n and j be easily determined? Let us start by considering the data for n = 6. The excedance program gives us the following list for N(6,j,k):

Excedance (j)	Weight (k)	Total (N(6,j,k))	Excedance (j)	Weight (k)	Total (N(6,j,k))
0	0	1	3	6	115
			3	7	69
1	1	31	3	8	68
1	2	15	3	9	32
1	3	7	3	10	14
1	4	3	3	11	3
1	5	1	3	12	1
2	3	115	4	10	31
2	4	69	4	11	15
2	5	68	4	12	7
2	6	32	4	13	3
2	7	14	4	14	1
2	8	3			
2	9	1	5	15	1

The data falls nicely into blocks by number of excedances, with the weights progressing downwards in increments of one in each separate block (except the very first and very last block). Looking at the permutation in S(6,3,12), 124563, we see that the three excedances are in positions 3, 4 and 5, the top three positions in which excedances are possible. For S(6,4,14), 134562, the excedances are w_2, w_3, w_4, and w_5, numbers in the top 4 possible positions. We would expect a similar pattern to be found at any j's highest weight. Similarly, we could expect that the lowest possible weight would have excedances in the first j positions. We see that for 345126 $\in$ S(6,3,6) this is indeed true. But what about the weights in between the highest and lowest?

Proposition 2: The weights k for S(n,j) are exactly the sequence

$$c_m = \left(\sum_{i=1}^{j}(n-i)\right) - m,\ m = 0, \ldots, (n-j-1) \bullet j.$$

Proof: The highest weight will occur when w has excedances in the j positions below n, that is, positions n-j, ..., n-2, n-1. Adding these j highest possible excedance positions, we get $\sum_{i=1}^{j} n-i = c_0$. By Proposition 1, position n-j-1 is fixed, so if we swap w_{n-j} with w_{n-j-1}, position n-j will give a non-excedance and position n-j-1 will give an excedance; furthermore the weight has dropped by 1. Inductively, we can always find an excedance preceded by a non-excedance; simply shift the excedance down one position and the total weight drops by one. If w has weight c_m, then this procedure produces a new permutation with weight $c_m-1 = c_{m+1}$. Shift the first excedance down the permutation until it is in the first position, then move on to the second excedance until it is in the second position, etc. When the j excedances are in the positions 1, 2, ..., j, there will be no more non-excedances followed by excedances and we reach the smallest possible weight, $\sum_{i=1}^{j} i$. Thus, we can find at least one possible way to get each weight k with $\sum_{i=1}^{j} i \le k \le \sum_{i=1}^{j} (n-i)$. ∎

Looking again at the data, it is immediately obvious that the sequence of values N(n,1,k) is $2^m - 1$, m = n-1, n-2, ..., 1. However, there appears to be no easy proof just looking at the expression "$2^m - 1$." But since every N(n,1,k) gives the same sequence, with the inclusion of one more term in each successive n, there could possibly be a connection from n to n + 1. For N(5,1,3) = 3, our permutations are 12435, 12534, 12543; for

$N(6,1,3) = 7$, our permutations are 124356, 125346, 125436, 126345, 126354, 126435, and 126453. Immediately, we see that there are twice-as-many-plus-one permutations for $n + 1$ than for n. If we remove the 6 from each permutation in $S(6,1,3)$ where 6 is at the end, we get exactly the permutations of $S(5,1,3)$: 12435, 12534, 12543. In the other permutations, if we swap position 3 (where the 6 is) and position 6 then again remove the 6, we get 12534, 12435, 12543, and the identity 12345. This same process works in general for $j = 1$.

Theorem 1:

$$N(n,1,k) = 2 \bullet N(n-1,1,k) + 1; \quad 1 \leq k \leq n-1.$$

Proof: *(i)* $N(n,1,k) \geq 2 \bullet N(n-1,1,k) + 1$: Let $m = N(n-1,1,k)$. Consider $\sigma \in S(n-1,1,k)$. Place n at the end, forming a permutation $\sigma n = w \in S_n$. Since $w_n = n$, the only excedance in w is in position k, as it was in the original permutation. Make a copy of the new permutation w; switch w_n with w_k, so that $k' = n$ and $n' = w_k$, to get w'. Since position n can never be exceeded and n always exceeds anywhere else but in position n, position k will hold the only excedance (see Example 1.1). Perform the same procedure on each of the m permutations in $S(n-1,1,k)$, getting $2 \bullet m$ permutations (the w's and the w''s). In addition, take the identity permutation, $12...n-1$. Place n at the end and then switch w_n and w_k as before. Position k will now be exceeded. And we have a total of $2 \bullet N(n-1,1,k) + 1$ permutations, each of

$\sigma = 12354 \in S(5,1,4)$
$w = 123546 \in S(6,1,4)$
$w' = 123645 \in S(6,1,4)$

Example 1.1

$w = 1253467 \in S(7,1,3) \mapsto \sigma = 125346 \in S(6,1,3)$
$w = 1274563 \in S(7,1,3) \mapsto \sigma = 123456 \in S(6,1,3)$
$w = 1273456 \in S(7,1,3) \mapsto \sigma = 126345 \in S(6,1,3)$

Example 1.2

which is an element of $S(n,1,k)$.

(ii) $N(n,1,k) \leq 2 \bullet N(n-1,1,k) + 1$: Choose any permutation $w \in S(n,1,k)$. The letter n is either in position n or position k (if it were elsewhere, that position would also have an excedance). If $w_n = n$, remove n to get $\sigma \in S_{n-1}$. The number which was an excedance in w is still an excedance in σ in the same position. There are no other excedances since there was only one to begin with thus $\sigma \in S(n-1,1,k)$. If $w_k = n$, swap w_n and w_k to obtain a permutation w' where $w_n' = n$ and $w_k' = w_n = g$ for some $g < n$. Remove n as before to get a permutation $\sigma \in S_{n-1}$ (see Example 1.2). I claim $g \geq k$. If $g < k$, some other position in addition to k would also be exceeded in $\sigma \in S_n$. If $w_n = g < k$, then w_g would have to be a number smaller than g since there is only one excedance. However, Proposition 1 tells us that all positions up to k are fixed, therefore $w_g = g$. Thus $g \geq k$. If $g = k$, our new permutation $\sigma \in S_{n-1}$ is the identity permutation (the "+ 1" of the equation). Otherwise, we have a permutation in $S(n-1,1,k)$. ∎

The sequence of numbers $N(n,1,k)$ now can be related to the sequence $2^m - 1$ by using the preceding theorem.

Corollary 1.1: $2^{n-k} - 1 = N(n,1,k)$ for $1 \le k \le n-1$.

Proof: The proof is by induction.

The first step is to show this statement is true for $n = 2$, $k = 1$: $2^{2-1} - 1 = 1 = N(2,1,1)$. Then, let us assume this is true for n and $1 \le k \le n-1$: $2^{n-k} - 1 = N(n,1,k)$. Finally, we must show this statement is true for $n+1$ and $k = 1, \ldots, n$.

$$\begin{aligned} N(n+1,1,k) &= 2 \bullet N(n,1,k) + 1 \text{ (Theorem 1)} \\ &= 2 \bullet (2^{n-k} - 1) + 1 \text{ for } 1 \le k \le n-1 \\ &= 2^{(n+1)-k} - 1. \end{aligned}$$

What about the case where $k = n$? Then we have

$$\begin{aligned} n &= \left(\sum_{i=1}^{1} [(n+1)-i]\right) - m, \text{ for some } m = 0, \ldots, (n-j-1)\bullet j \\ &= (n + 1) - 1 - m \\ n &= n - m. \end{aligned}$$

Thus $m = 0$ and $k = n = c_0$. Therefore $k = n$ is the highest possible weight, for which there is only one permutation (Proposition 2) and $N(n+1,1,n) = 1 = 2^{(n+1)-n} - 1$. ∎

Since the sequence $N(n,n-1,k)$ follows the same pattern, it would be nice if the same reasoning used in Theorem 1 could be used to prove the relationship between $N(n,n-1,k)$ and $N(n+1,n,k)$. Unfortunately, it cannot. However, a later theorem (Theorem 3) does explain the reasons behind the symmetry between $N(n,1,k)$ and $N(n,n-1,k)$.

When examining $N(n,j,k)$ for all the data (see Appendix A), one extremely interesting pattern is seen by comparing $N(n,j,k)$ with $N(n+1,j,k+j)$. The highest few totals from the lower block appear in the higher one. However, the entire excedance block does not carry through from n to n+1 but rather only a select few. For instance, consider the blocks for j = 2:

Excedance n=5	Weight	Total	Excedance n=6	Weight	Total
2	3	31	2	3	115
2	4	17	2	4	69
2	5	**14**	2	5	68
2	6	**3**	2	6	**32**
2	7	**1**	2	7	**14**
			2	8	**3**
			2	9	**1**

n=7			n=8		
2	3	391	2	3	1267
2	4	245	2	4	813
2	5	260	2	5	896
2	6	146	2	6	542
2	7	**99**	2	7	417
2	8	**32**	2	8	**209**
2	9	**14**	2	9	**99**
2	10	**3**	2	10	**32**
2	11	**1**	2	11	**14**
			2	12	**3**
			2	13	**1**

Notice that from n=5 to n=6, three terms carry over; from 6 to 7, four terms carry; and from 7 to 8, five terms carry. That is, n-j weights from one block are seen in the corresponding block for n+1. What could be so special about the permutations in these few weights? Let us consider some permutations from both carry-over and noncarry-over weights.

S(5,2,6):	S(5,2,4):	S(6,2,7):	S(6,2,6):
13254	21435	125634	132645
14253	41523	143265	213465
14352	52413	152463	412365
(carry)	(non)	(carry)	(non)

These representative permutations indicate that the carry-over weights' permutations always have a 1 in position 1 (hence a non-excedance) while the noncarry-over blocks may or may not have an excedance in position 1. In fact, there are no permutations with excedances in position 1 in the first n-j-1 weights in any block.

Lemma 1: Let $1 \le j < n-1$. Then the permutations in $S(n,j,k)$, as k ranges through the highest n-j-1 weights, do not have an excedance in position 1. However, there may be an excedance in position 1 for c_m with $n-j-1 \le m \le (n-j-1) \bullet j$.

Proof: We fix n and j and consider the weights c_m, $m = 0, \ldots, n-j-2$. For the highest weight, $w \in S(n,j,c_0)$ is of the form *n...ne...en*, where there are j *e*'s (*n* indicates a non-excedance, *e* an excedance). The first *e* is not in position 1 unless $j = n-1$ (which does not hold). The form of a permutation with the largest possible weight without a 1 in position one is *nen...ne...en*. Why? If the top j-1 excedances were not in the highest j-1 possible excedance places the first excedance could be shifted to position 1 and whichever excedance preceded a non-excedance could be shifted up, and the weight would stay the same. However, with this configuration, it is impossible to shift the j excedances and put one of them in position 1. This permutation's weight is $\sum_{i=1}^{j-1}(n-i) + 2$. I claim this is c_{n-j-2}.

We have:

$$\begin{aligned} c_{n-j-2} &= \sum_{i=1}^{j}(n-i) - (n-j-2) \\ &= (n-1) + \ldots + (n-j+1) + (n-j) - (n-j) + 2 \\ &= (n-1) + \ldots + (n-j+1) + 2 \end{aligned}$$

$$= \sum_{i=1}^{j-1} (n-i) + 2.$$

The largest possible k with an excedance in position 1 would be of the form *en...nne...en* (an excedance, followed by n-j-1 non-excedances, j-1 more excedances, then another non-excedance). The weight of this permutation is $\sum_{i=1}^{j-1}(n-i) + 1$. I claim this weight is precisely c_{n-j-1}.

$$\begin{aligned} c_{n-j-1} &= \sum_{i=1}^{j}(n-i) - (n-j-1) \\ &= (n-1) + \ldots + (n-j+1) + (n-j) - (n-j) + 1 \\ &= (n-1) + \ldots + (n-j+1) + 1 \\ &= \sum_{i=1}^{j-1}(n-i) + 1. \end{aligned}$$

We know that the smallest weight, $c_{(n-j-1)\bullet j} = \sum_{i=1}^{j} i$ occurs when w = *e...en...n* which obviously has an excedance in position 1, as was noted in Proposition 2. By Proposition 2, we know that it is possible to find each weight between c_{n-j-1} and the lowest possible weight by simply moving every excedance (other than the one already in position 1) down one position until they are all lined up at the beginning of the permutation. Thus each set $S(n,j,c_i)$, $i \leq n-j-1$, will contain some permutation(s) with an excedance in position 1. ∎

Now we know that the idea of carryover has something to do with position one, but what? Consider 13254 ∈ S(5,2,6) and 124365 ∈ S(6,2,8). Looking at them in another way, $\substack{13254 \\ 124365}$, notice that each letter differs by one. That is, if we add one to each letter in 13254 and then place a 1 at the front, we produce 124365. This same process works for every one of the carryover permutations.

Theorem 2: $N(n,j,k) = N(n+1,j,k+j)$ for all n and j and the top n-j-2 values of k.

Proof: Let us define a map ϕ: $S_n \Rightarrow S_{n+1}$, given by $\phi(w) = \sigma$ where $\sigma_1 = 1$, $\sigma_i = w_{i-1} + 1$, $(i = 2, \ldots, n)$. In other words, σ is obtained by adding one to each letter and then placing a 1 at the beginning of the new permutation. I claim that if $w \in S(n,j,k)$, then $\phi(w) \in S(n+1,j,k+j)$.

Let $w \in S(n,j,k)$. By Proposition 1, we know that up to the first exceeded position all letters are fixed, i.e. $w_1 = 1$, $w_2 = 2$, etc. Apply ϕ; see Example 2.1. Every letter increases by 1 and its position is now one greater. Any fixed point in w is still fixed in σ, a non-excedance is still a non-excedance, and each of the j excedances is still an excedance; everything is just one position higher. Thus, the weight of σ increases by j (adding 1 for each of the j excedances of w). Therefore, $\sigma \in S(n+1,j,k+j)$.

$$
\begin{aligned}
w &= \ \ 124563 \in S(6,3,12) \\
&\ \ \ \ \ 235674 \\
\sigma &= 1235674 \in S(7,3,15)
\end{aligned}
$$

Example 2.1

To reverse this process, let $\sigma \in S(n+1,j,k+j)$. We can recover w from σ as follows: subtract 1 from each letter to get σ'. If $\sigma_i = 1$ then σ_{i-1}' will equal 0 and thus drop out. Shift every succeeding letter one position to the left and remove position n + 1 (which

$$
\begin{aligned}
\sigma &= 124635 \in S(6,2,7) \\
&\ \ \ \ 013524 \\
w &= \ \ 13524 \in S(5,2,5)
\end{aligned}
$$

Example 2.2

will now be empty) to obtain w (see Example 2.2). Obviously, $w \in S_n$. Due to the restrictions on k, i = 1 so $\sigma_1 = 1$ (Lemma 1) and thus <u>every</u> letter decreases its position by 1. Every letter in σ is as it was in w (staying fixed, non-exceeding, or exceeding), but one position lower. So, the weight drops by j (subtract 1 for each excedance in w) and therefore $w \in S(n,j,k)$. ∎

Finally, we get to one of the most obvious symmetries, but the hardest to prove: Why are the numbers N(n,j,k) symmetric with respect to excedance blocks?

Looking at an example from the previous data, we see that N(6,1,3) = N(6,4,12). A permutation from S(6,1,3) is 124356; from S(6,4,12), 231564. Notice that 124356 has an excedance only in position 3 while that is the only position (other than n) that is not exceeded in 231564.

Or consider $124365 \in S(6,2,8)$ and $143562 \in S(6,3,11)$. In the first permutation, excedances occur in positions 3 and 5 while in the second permutation the excedances are in positions 2, 4, and 5. Non-excedances are found in 1, 2 and 4 for the former and in 1 and 3 in the latter (and the inconsequential position 6). Thus, the first five numbers in each permutation are of the form *nnene* and *nenee*. Look closely and you can see that these are reverse mirror images. That is, one is the other backwards with *n*'s and *e*'s switched. This same unusual pattern is found throughout all the excedance blocks, and is the basis for the proof of the following theorem.

Theorem 3: $N(n,j,k) = N(n,n-j-1,k')$ where k, k' range from the highest to lowest weights for their respective j number of excedances.

σ: 132654 $\in S(6,2,6)$
σ': 562314
σ'': 215463 $\in S(6,3,9)$

Example 3.1

Proof: Let $\sigma \in S(n,j,k)$. Reverse the order of the first n-1 positions, leaving σ_n as is, to get σ'. Next, take the complement of σ' with respect to n+1, getting σ''. That is, $\sigma = \sigma_1\sigma_2\ldots\sigma_{n-1}\sigma_n$, $\sigma' = \sigma_{n-1}\sigma_{n-2}\ \ldots\ \sigma_2\sigma_1\sigma_n$ and $\sigma'' = (n+1 - \sigma_{n-1})\ \ldots\ (n+1 - \sigma_{n-i})\ \ldots\ (n+1 - \sigma_n)$ (see Example 3.1).

i) If $\sigma_m > m$ in σ, I claim $\sigma''_{n-m} \leq n-m$. In σ', σ_m is in position $n-m$. In σ'', σ_{n-m} contains $n+1 - \sigma_m$. We now show that $n+1 - \sigma_m \leq n-m$.

If $\sigma_m = m+1$, $n+1-(m+1) = n-m$.

If $\sigma_m > m+1$, $n+1-\sigma_m < n+1 - (m+1) = n-m$.

Thus, an excedance place in σ leads us to a fixed point or a non-excedance in σ''.

ii) If $\sigma_m \leq m$ in σ, I claim $\sigma''_{n-m} > n-m$.

If $\sigma_m = m$, $\sigma''_{n-m} = n+1-\sigma_m = n+1-m > n-m$.

If $\sigma_m < m$, $\sigma''_{n-m} = n+1-\sigma_m > n+1-m > n-m$.

Thus, a non-excedance or a fixed point in σ leads us to an excedance place in σ''.

On the other hand,

i) if $\sigma''_{n-m} \leq n-m$, I claim $\sigma_m > m$.

$$\sigma''_{n-m} = n+1 - \sigma_{n-(n-m)} = n+1 - \sigma_m \leq n-m$$

$$n - \sigma_m \leq n-m - 1$$

$$\sigma_m \geq m+1$$

$$\sigma_m > m.$$

Thus, a non-excedance or a fixed point place in σ'' comes through the procedure from an excedance place in σ.

ii) If $\sigma''_{n-m} > n-m$, I claim $\sigma_m \le m$

$$\sigma''_{n-m} = n+1 - \sigma_{n-(n-m)} = n+1 - \sigma_m > n-m$$

$$n - \sigma_m > n-m - 1$$

$$\sigma_m < m+1$$

$$\sigma_m \le m.$$

Therefore, we see that an excedance place in σ'' comes through the procedure from a non-excedance or a fixed point place in σ.

Now, all that is left to show is that these reversals actually land us in the proper places for excedances and so give the correct k'. In other words, we want to show that if a permutation in $S(n,j,k)$ has an excedance in position i, that the corresponding permutation in $S(n,n-j,k')$ has a non-excedance in the "swapped" position, and so forth.

First, from Proposition 2 we know that the least possible $k = 1 + 2 + \ldots + j$ and the least $k' = 1 + 2 + \ldots + (n-1-j)$. Also from Proposition 2, it is obvious that $k' > k$ for $j < (n-1)/2$. And so each pair k, k' differ by a constant. We have

$$\begin{aligned} k - k' &= (1 + 2 + \ldots + n-1-j) - (1 + 2 + \ldots + j) \\ &= (j+1) + (j+2) + \ldots + (n-1-j) \\ &= (n^2 - n - 2nj) / 2. \end{aligned}$$

So $k' = (n^2 - n - 2nj / 2) - k$.

Let $e_1, e_2, \ldots, e_j$ be the excedance places for σ. Let $f_1, f_2, \ldots, f_{n-j}$ be the non-excedance places for σ. Recall that

$f_{n-j} = n$, thus $n - f_1$, $n-f_2$, ... , $n-f_{n-j-1}$ are the excedance places for σ''.

The sum of all excedance places in σ is $\sum_{i=1}^{j} e_i = k$.

The sum of all positions is $\sum_{i=1}^{n} i = n(n+1) / 2$.

So the sum of all non-excedance places in σ'' is

$$\sum_{i=1}^{n-j} f_i = (n(n+1) / 2) - k,$$

and $\sum_{i=1}^{n-j-1} f_i = [n(n+1) / 2] - k - n$.

And therefore the sum of all excedance places in σ'' is

$$\begin{aligned}\sum_{i=1}^{n-j-1} (n - f_i) &= n(n-j-1) - (n(n+1) / 2) - k - n) \\ &= (n^2 - n - 2nj) / 2 + k \\ &= k'. \quad \blacksquare\end{aligned}$$

When considering properties of permutations, one must always consider an extension of theorems to multisets. Multisets are sets of numbers where one or more letters are repeated; for instance, {1, 1, 1, 2, 3, 3} is a multiset of six letters. Each letter is usually considered non-distinct, thus switching the first and second letters does not produce a distinct permutation. Using typical multiset notation, {1, 1, 1, 2, 3, 3} is $\{1^3 2^1 3^2\}$, where the superscript, or exponent, indicates the number of copies of that letter. In general, we have $\{1^{p1} 2^{p2} \ldots n^{pn}\}$ where each pi is some number 0, ..., n. The type of a permutation is the set of exponents, {p1 p2 ... pn}. Multisets have fewer total permutations than the numbers {1, 2, ..., n} due to these repeated letters.

The distribution of descent weights on multisets produces a very nice pattern which is related to the Mahonian distribution of the major index. From the previously shown work, I figured that excedance weights on multisets would not produce the same distribution as descents, but I wondered if the corresponding bivariate distribution would be similar to the Paulukonis distribution.

Before I could look at the bivariate distribution on multisets, I had to decide what the set of positions would be. For multiset descent weights, the same positions are used as are used for regular permutation sets, that is the set {1, 2, ..., n}, where n is the total number of letters in the set. The other choice of position numbering is the identity permutation for that multiset. For instance, if the multiset is $\{1^2 2^1 3^3\}$, the six positions are number 112333. I chose the identity permutation first; results from running 5-letter and 6-letter multisets of the numbers 1, 2, and 3 through the computer program are shown in Appendix C. These seem inconclusive; so I also processed the permutations using {12...n} as the position place (see Appendix D). I later learned that the multiset identity would be preferred, but neither version seems particularly preferable on a first analysis. In fact, I believe that the second set of data has more interesting properties than the first.

A preliminary analysis of both sets of data leads to a few conjectures. Since there are so many different sets that are n letters long, our previous notation needs to be modified for multisets. If $\sigma = 1^{p1}2^{p2}\ldots n^{pn}$, then $S(\sigma,j,k)$ is the set of permutations on the multiset $\{1^{p1}2^{p2}\ldots n^{pn}\}$ that have j excedances summing to weight k. The rest of the notation follows hence.

Dealing first with the distribution using the multiset identity permutation as positions, I conjecture that:

A. For permutations with n-1 copies of one letter *a* and 1 of another letter *b*, $N(a^{n-1}b^{1},j,k) = N(a^{n-1}b^{1},1,k) = n-1$ where k is the smaller of the two letters.

B. Any permutations that have p of any one letter and q of another letter will have the same sequence of $N(\sigma,j,k)$; the weights may vary but the range of the number of excedances will be the same. For example, in the multisets of 5 letters, all words with 2 of one letter and 3 of another letter have totals of 6 and 3. In cases of three distinct letters, there may be a third letter that has a fixed number r of repetitions (r = 1...) in all words. This works in selected cases. For example, those 6-letter multisets of $\{1^2 2^3 3^1\}$ and $\{1^2 2^1 3^3\}$ have totals of 14, 3, 15, 18, and 9. The 1 is fixed at two repetitions while the 2 and 3 vary between 1 and 3 repetitions.

For multisets weighted by the set {12...n} as position, I conjecture that:

A. The only multisets that will have any permutations in $S(n,0,0)$ are those where there is at least one 1.

B. For permutations with n-1 copies of one letter m and 1 of another letter p, $m < p$, $N(n,j,k) = n-1$ where $j = 0$, $k = 0$ if $m = 1$ and $j = 1$, $k = 1$ if $m > 1$. Also, $N(n,m,p) = 1$. If there are n-1 copies of p and only one of m, then the totals are reversed.

C. Let σ be of type ω; if σ' is of type ω' where ω' is the reverse of ω, then the set of values of $N(\sigma,j,k)$ is the same as the set of values of $N(\sigma',j,k)$, not necessarily in the same order. For example, $\{1^3 2^2 3^1\}$, of type $\{321\}$ has the set of values of $N(1^3 2^2 3^1,j,k) = \{24, 26, 6, 4\}$. The opposite type is $\{123\}$, which corresponds to the set $\{1^1 2^2 3^3\}$. The set of values of $N(1^1 2^2 3^3,j,k) = \{4, 26, 6, 24\}$.

The problem with looking at excedances on multiset permutations is that we are dealing with a larger variety within a certain length. Additionally, whereas a descent is a descent no matter how one chooses to weight it, an excedance depends completely upon what the position numbering is. It is not clear if there are more patterns or if the patterns already noticed will be reinforced; clearly, more research is needed.

APPENDIX A
PAULUKONIS DISTRIBUTION

The totals for TWO are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	1

The totals for THREE are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	3
1	2	1
2	3	1

The totals for FOUR are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	7
1	2	3
1	3	1
2	3	7
2	4	3
2	5	1
3	6	1

The totals for FIVE are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	15
1	2	7
1	3	3
1	4	1
2	3	31
2	4	17
2	5	14
2	6	3
2	7	1
3	6	15
3	7	7
3	8	3
3	9	1
4	10	1

The totals for SIX are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	31
1	2	15
1	3	7
1	4	3
1	5	1
2	3	115
2	4	69
2	5	68
2	6	32
2	7	14
2	8	3
2	9	1
3	6	115
3	7	69
3	8	68
3	9	32
3	10	14
3	11	3
3	12	1
4	10	31
4	11	15
4	12	7
4	13	3
4	14	1
5	15	1

The totals for SEVEN are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	63
1	2	31
1	3	15
1	4	7
1	5	3
1	6	1
2	3	391
2	4	245
2	5	260
2	6	146
2	7	99
2	8	32
2	9	14
2	10	3
2	11	1
3	6	675
3	7	445
3	8	522
3	9	385
3	10	219
3	11	105
3	12	47
3	13	14
3	14	3
3	15	1
4	10	391
4	11	245
4	12	260
4	13	146
4	14	99
4	15	32
4	16	14
4	17	3
4	18	1
5	15	63
5	16	31
5	17	15
5	18	7
5	19	3
5	20	1
6	21	1

The totals for EIGHT are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	127
1	2	63
1	3	31
1	4	15
1	5	7
1	6	3
1	7	1
2	3	1267
2	4	813
2	5	896
2	6	542
2	7	417
2	8	209
2	9	99
2	10	32
2	11	14
2	12	3
2	13	1
3	6	3451
3	7	2385
3	8	3014
3	9	2531
3	10	1948
3	11	1094
3	12	699
3	13	296
3	14	136
3	15	47
3	16	14
3	17	3
3	18	1
4	10	3451
4	11	2385
4	12	3014
4	13	2531
4	14	1948
4	15	1094
4	16	699
4	17	296
4	18	136
4	19	47
4	20	14
4	21	3
4	22	1
5	15	1267
5	16	813
5	17	896
5	18	542
5	19	417
5	20	209
5	21	99
5	22	32
5	23	14
5	24	3
5	25	1
6	21	127
6	22	63
6	23	31
6	24	15
6	25	7
6	26	3
6	27	1
7	28	1

APPENDIX B
DISTRIBUTION PROGRAM

```
#include <stdio.h>
#include <stdlib.h>

int exced,
        weight,
        j,
        k,
        length,
        eof,
        perm[20],
        total[100][100];
FILE *f;


find_exced () {


for (j=0; j < length; j++) {
        if (perm[j] > (j+1)) {
                exced++;
                weight = weight + j + 1;
        }
}

total[exced][weight]++;

printf ("for the permutation ");
for (j = 0; j < length; j++)
        printf ("%d", perm[j]);
printf ("\nexcedences are %d;  weight is %d \n", exced, weight);
weight = 0;
exced = 0;

} /* end find */



main () {

perm[0] = 1;

length = 5;
weight = 0;
exced = 0;
eof = 0;

for (j =0; j <= 100; j++) {
        for (k = 0; k <=100; k++) {
                total[j][k] = 0;
        }
}


f = fopen("perms5.txt", "r");
if (f == NULL)
        return;
while (eof == 0) {
        if (fscanf (f, "{%i, %i, %i, %i, %i}, ", &perm[0], &perm[1], &perm[2], &p
                                eof = 1;
                } /*end if*/
        if (eof == 0)
                find_exced ();

} /* end while */

printf ("The totals are: \n");
printf ("   EXCEDENCE    WEIGHT  TOTAL\n");
for (j = 0; j <= 100; j++) {
        for (k = 0; k <= 100; k++) {
                if (total[j][k] != 0) {
                        printf ("\t %d \t %d \t%d \n", j, k, total[j][k]);
                }
        }
}


} /* end main */
```

APPENDIX C

MULTISET DISTRIBUTION WITH MULTISET IDENTITY AS POSITIONS

(LENGTH = 5, LETTERS = {1, 2, 3})

The totals for 11112 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	4

The totals for 11113 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	4

The totals for 11122 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	6
2	2	3

The totals for 11123 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	9
1	2	1
2	2	6
2	3	3

The totals for 11133 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	6
2	2	3

The totals for 11222 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	6
2	2	3

The totals for 11223 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	10
1	2	2
2	2	7
2	3	8
3	4	2

The totals for 11233 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	10
1	2	2
2	2	7
2	3	8
3	4	2

The totals for 11333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	6
2	2	3

The totals for 12222 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	4

The totals for 12223 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	7
1	2	3
2	3	9

The totals for 12233 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	8
1	2	4
2	3	14
2	4	1
3	5	2

The totals for 12333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	7
1	2	3
2	3	9

The totals for 22223 are:

EXCEDANCE	WEIGHT	TOTAL
1	2	4

The totals for 22233 are:

EXCEDANCE	WEIGHT	TOTAL
1	2	6
2	4	3

The totals for 22333 are:

EXCEDANCE	WEIGHT	TOTAL
1	2	6
2	4	3

The totals for 23333 are:

EXCEDANCE	WEIGHT	TOTAL
1	2	4

please note every set also contains

0	0	1

from the identity permutation

MULTISET DISTRIBUTION WITH MULTISET IDENTITY AS POSITIONS

(LENGTH = 6, LETTERS = {1, 2, 3})

The totals for 111112 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	5

The totals for 111113 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	5

The totals for 111122 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	8
2	2	6

The totals for 111123 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	12
1	2	1
2	2	12
2	3	4

The totals for 111133 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	8
2	2	6

The totals for 111222 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	9
2	2	9
3	3	1

The totals for 111223 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	15
1	2	2
2	2	21
2	3	12
3	3	3
3	4	6

The totals for 111233 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	15
1	2	2
2	2	21
2	3	12
3	3	3
3	4	6

The totals for 111333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	9
2	2	9
3	3	1

The totals for 112222 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	8
2	2	6

The totals for 112223 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	14
1	2	3
2	2	15
2	3	18
3	4	9

The totals for 112233 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	16
1	2	4
2	2	19
2	3	28
2	4	1
3	4	16
3	5	4
4	6	1

The totals for 112333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	14
1	2	3
2	2	15
2	3	18
3	4	9

The totals for 113333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	8
2	2	6

The totals for 122222 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	5

The totals for 122223 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	9
1	2	4
2	3	16

The totals for 122233 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	11
1	2	6
2	3	30
2	4	3
3	5	9

The totals for 122333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	11
1	2	6
2	3	30
2	4	3
3	5	9

The totals for 123333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	9
1	2	4
2	3	16

The totals for 133333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	5

The totals for 222223 are:

EXCEDANCE	WEIGHT	TOTAL
1	2	5

The totals for 222233 are:

EXCEDANCE	WEIGHT	TOTAL
1	2	8
2	4	6

The totals for 222333 are:

EXCEDANCE	WEIGHT	TOTAL
1	2	9
2	4	9
3	6	1

The totals for 223333 are:

EXCEDANCE	WEIGHT	TOTAL
1	2	8
2	4	6

The totals for 233333 are:

EXCEDANCE	WEIGHT	TOTAL
1	2	5

please note every set also contains

0 0 1

from the identity permutation

APPENDIX D

MULTISET DISTRIBUTION WITH {1,2,...,N} AS POSITIONS

(LENGTH = 5, LETTERS = {1, 2, 3})

The totals for 11112 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	4
1	1	1

The totals for 11113 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	3
1	1	1
1	2	1

The totals for 11122 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	6
1	1	4

The totals for 11123 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	9
1	1	7
1	2	3
2	3	1

The totals for 11133 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	3
1	1	3
1	2	3
2	3	1

The totals for 11222 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	4
1	1	6

The totals for 11223 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	9
1	1	15
1	2	3
2	3	3

The totals for 11233 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	6
1	1	12
1	2	6
2	3	6

The totals for 11333 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	3
1	2	3
2	3	3

The totals for 12222 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	4

The totals for 12223 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	3
1	1	13
1	2	1
2	3	3

The totals for 12233 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	3
1	1	15
1	2	3
2	3	9

The totals for 12333 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	7
1	2	3
2	3	9

The totals for 13333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	1
1	2	1
2	3	3

The totals for 22223 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	4
2	3	1

The totals for 22233 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	6
2	3	4

The totals for 22333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	4
2	3	6

The totals for 23333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	1
2	3	4

MULTISET DISTRIBUTION WITH {1,2,...,N} AS POSITIONS

(LENGTH = 6, LETTERS = {1, 2, 3})

The totals for 111112 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	5
1	1	1

The totals for 111113 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	4
1	1	1
1	2	1

The totals for 111122 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	10
1	1	5

The totals for 111123 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	16
1	1	9
1	2	4
2	3	1

The totals for 111133 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	6
1	1	4
1	2	4
2	3	1

The totals for 111222 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	10
1	1	10

The totals for 111223 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	24
1	1	26
1	2	6
2	3	4

The totals for 111233 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	18
1	1	22
1	2	12
2	3	8

The totals for 111333 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	4
1	1	6
1	2	6
2	3	4

The totals for 112222 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	5
1	1	10

The totals for 112223 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	16
1	1	34
1	2	4
2	3	6

The totals for 112233 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	18
1	1	42
1	2	12
2	3	18

The totals for 112333 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	8
1	1	22
1	2	12
2	3	18

The totals for 113333 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	4
1	2	4
2	3	6

The totals for 122222 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	5

The totals for 122223 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	4
1	1	21
1	2	1
2	3	4

The totals for 122233 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	6
1	1	34
1	2	4
2	3	16

The totals for 122333 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	4
1	1	26
1	2	6
2	3	24

The totals for 123333 are:

EXCEDANCE	WEIGHT	TOTAL
0	0	1
1	1	9
1	2	4
2	3	16

The totals for 133333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	1
1	2	1
2	3	4

The totals for 222223 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	5
2	3	1

The totals for 222233 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	10
2	3	5

The totals for 222333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	10
2	3	10

The totals for 223333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	5
2	3	10

The totals for 233333 are:

EXCEDANCE	WEIGHT	TOTAL
1	1	1
2	3	5

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